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ANALYSIS OF THE PARADOX OF THE INTERACTION OF A VORTEX FILAMENT WITH A PLANE*

M.A. GOL'DSHTIK and V.N. SHTERN

It will be shown that the crisis occurring in the interaction of a vortex filament with a plane perpendicular to it, consisting of the failure of the solution to exist at a finite Reynolds number, is due to the formation of an extremely intense induced jet in the axial region. External and internal expansions are constructed for the near-critical situation, yielding an exhaustive characterization of the structure of the solution. It is observed that if the circulation is specified not on the axis but on a finite-angled cone, the solution continues to exist at all Reynolds numbers. The vortex filament is considered as a limiting case of a cone.

The interaction of a vortex filament with a plane has been studied in many publications, beginning with /1/, where a paradoxical fact was established: a solution with bounded meridian velocity exists at Reynolds numbers not exceeding a certain critical value, ceasing to exist at higher Reynolds numbers. A detailed exposition of these results and a survey of the literature up to 1980 can be found in /2/. As shown in this paper, the source of the paradox is that the axis of symmetry lies in the flow region. The motion under consideration may be interpreted as being generated by a rotating needle at right angles to the plane. In this case, however, since the dimensions of the needle are finite, its immediate vicinity is a region of non-selfsimilarity, which remains outside the scope of the discussion.

Serrin /3/ assumed that the longitudinal component of the velocity has a logarithmic singularity along the axis, treating the coefficient of the logarithm as a parameter in addition to the circulation. Serrin showed that the plane of these parameters contains a curve which bounds the region of existence of the solutions of the class in question. The value of the coefficient of the logarithm was determined by postulating an additional hypothesis of a phenomenological nature.

In this paper a different approach is taken. The flow nucleus is situated in a small-angled cone, on whose surface the circulation and other appropriate boundary conditions are given. The angle is then allowed to approach zero. The longitudinal component of the velocity of the external flow remains bounded, but in the critical situation a singularity forms along the axis - a linear sink of well-defined strength. At supercritical Reynolds numbers the passage to the limit of a vortex filament produces the same external flow, identical with the critical flow.

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1. Statement of the problem and basic equations. We consider the steady axisymmetric flow of a viscous incompressible liquid, generated by a semi-infinite vortex filament of given circulation $2\pi\Gamma_1$. The filament rests against a fixed plane, on which the adherence conditions must hold. A generalization of this situation is a vortex-sink combination flow on an impermeable cone with half-angle θ_1.

In a spherical system of coordinates r, θ, φ with a pole at the end of the vortex filament or the apex of the cone, the velocity field corresponding to a selfsimilar solution of the problem may be written as

$$v_r = -\frac{\nu}{r} y'(x), \quad v_\theta = -\frac{\nu y(x)}{r\sin\theta}, \quad v_\varphi = \frac{\nu\Gamma(x)}{r\sin\theta}, \quad x = \cos\theta \tag{1.1}$$

Here $y(x)$ is an unknown non-dimensional function, related to the stream function Ψ by the equation $\Psi = \nu r y$ and $\Gamma(x)$ is the unknown non-dimensional circulation, related to the given (dimensional) circulation by the equation $\Gamma_1 = \nu\Gamma(x_1) = \nu\mathrm{Re}$, where $\mathrm{Re} = \Gamma_1/\nu$ is the Reynolds number.

Since in this formulation of the problem no reference length, only Γ_1 - which has the dimensions of kinematic viscosity - is given, the representation (1.1) must be selfsimilar: if a solution exists, it is selfsimilar /4/. Substitution of (1.1) into the Navier-Stokes equations (see, e.g., /2/) gives

$$(1-x^2)^2 y'''' - 4x(1-x^2) y''' = {}^1/_2 (1-x^2)(y^2)''' + 2\Gamma\Gamma' \tag{1.2}$$

$$(1-x^2)\Gamma'' = y\Gamma' \tag{1.3}$$

Thus, the entire variety of problems in this selfsimilar class reduces to the solution of system (1.2), (1.3) with appropriate boundary conditions.

The most general boundary-value problem is obtained if one sets boundary conditions on the conical surfaces $x = x_1$ and $x = x_2, -1 \leqslant x_2 < x_1 \leqslant 1$. For example, one might specify the velocity vector on cones $x_1, x_2 \neq \pm 1$ in the context of (1.1). In that case the coefficients of Eqs.(1.2), (1.3) have no singularities in the interval of integration. And although the general theorems of existence for steady-state solutions /5, 6/ do not apply to the selfsimilar class (1.1), there is no doubt that the problem is solvable - at least, for small Reynolds numbers. When one of the semi-axes lies in the flow region, the coefficients of the highest-order derivatives vanish at $|x| = 1$. As will be shown later, this circumstance may radically affect the properties of the boundary-value problem and its solvability.

In the situation under discussion there are two alternative approaches: one either treats the axis as the interior of the flow region, in which can analyticity conditions are formulated on it; or one continues to view the axis as the flow boundary, containing the source of motion, in which case one has a well-defined singularity along the axis. A distinct case is the problem of an unbounded region in which the flow is generated by a source of momentum at the origin (Landau jet /7/). Thus, the flow may be determined either by boundary conditions or by a point source. The two cases, incidentally, are mutually exclusive.

However formulated, the problem is clearly over-determined, which is rather at variance with one's intuitive concepts of the independence and compatibility of such sources of motion in real jets. Indeed, for a jet emerging from a hole in a wall one can independently determine both the momentum flux from the hole and the velocity field at the wall, such as adherence conditions. However, it turns out that this cannot be done in the limiting case of a infinitely small hole; indeed, by a theorem of Sedov, the solution must be selfsimilar and belong to class (1.1), but since the problem is over-determined no such solution exist. This means that, apart from Landau's solution, there are no selfsimilar flows generated by a point source of momentum. But then it is natural to consider the jets induced by the motion of the boundaries as induced jets.

2. The properties of the solutions and critical phenomena. It will be convenient to define a function F as follows:

$$F''' = 2\Gamma\Gamma'/(1-x^2) \tag{2.1}$$

Then Eq.(1.2) can be integrated three times:

$$(1-x^2) y' + 2xy - {}^1/_2 y^2 = F + C_3 + C_2 x - C_1 x^2 \tag{2.2}$$

Eq.(1.3) can also be integrated:

$$\Gamma = \int \exp\left\{\int 2S dx\right\} dx, \quad S = {}^1/_2 y/(1-x^2) \tag{2.3}$$

where, by (2.2), $S(x)$ satisfies the equation

$$S' - S^2 = {}^1/_2\,(F + C_3 + C_2x - C_1x^2)/(1 - x^2) \equiv \Phi(x) \tag{2.4}$$

The paradoxical properties observed in solutions of the class under consideration are due to the fact that Eq.(2.4) with the above function $\Phi(x)$ is the canonical Ricatti equation, and therefore $S(x)$ may have poles even in regions where the right-hand side is continuous. If variation of the parameters brings a pole into the range of allowable values of x, the solution of the boundary-value problem ceases to exist. As the pole approaches this range, boundary layers are formed.

In this paper we concentrate our attention on the critical phenomena occurring when the pole crosses the end of the range at finite Reynolds numbers. To be precise, we have in mind a situation in which all quantities in the boundary-value problem remain bounded, while the solution itself becomes unbounded. It turns out that a crisis of this kind is possible only if the axis $|x| = 1$ falls in the region of integration.

Considering the case in which the boundary condition at $x = x_1$ is the impermeability condition $y(x_1) = 0$, let us convince ourselves that the pole cannot cross the boundary if $|x_1| \neq 1$. Substituting $S = -T'/T$ into Eq.(2.4), we obtain $T'' + \Phi(x)\,T = 0$. If $\Phi(x)$ is bounded, it follows from Sturm's Theorem that the roots of $T(x)$ and $T'(x)$ alternate and are simple. Indeed, if T and T' vanish at one and the same x, then $T \equiv 0$. Consequently, the zeros and poles of the function $y(x)$ have the same properties.

Now a pole can cross the boundary $x = x_1$, i.e., a pole and a root can coincide, only if $\Phi(x)$ becomes infinite at $x = x_1$. Since $|x_1| \neq 1$, the numerator of expression (2.4), which characterizes the sources of the motion must become infinite. Thus, if the sources of the motion, as determined by the boundary conditions, are bounded, the same is true of the solution.

This is not so if the axis falls in the flow region. In that case a crisis may occur at a finite Reynolds number, as can be shown by considering a few known exact analytic solutions.

Let us consider a typical case for the class (1.1) - the problem of a sink in a plane. Suppose that a region filled with a visous liquid contains a plane made of material contracting without rotation to a centre, which is a sink of strength $2\pi Q$, so that in the plane $v_r = -Q/r,\ v_\theta = v_\varphi = 0$. The liquid motion induced by the contracting plane is necessarily selfsimilar and of class (1.1). On the axis $x = 1$ we impose the analyticity conditions: $y(1) = 0,\ y'(1),\ y''(1)$ are bounded, and using (2.2) we find that $C_2 = 2C_1,\ C_3 = -C_1$. Consequently, Eq.(2.2) may be rewritten in this case as

$$(1 - x^2)\,y' + 2xy = {}^1/_2 y^2 - C_1\,(1 - x)^2 \tag{2.5}$$

The constant C_1 is determined from (2.5) and the condition $y(0) = 0$. By (1.1), we have

$$C_1 = -y'(0) = -Q/\nu \equiv -\mathrm{Re}$$

For $\mathrm{Re} > {}^1/_2$ the solution of Eq.(2.5) is

$$y = \frac{2\,\mathrm{Re}\,(1 - x)}{\gamma\,\mathrm{ctg}\,[{}^1/_2\gamma\ln(1 + x)] - 1}, \quad \gamma = \sqrt{2\,\mathrm{Re} - 1} \tag{2.6}$$

This solution is actually due to Squire /8/, though in the context of a different interpretation. For a value of γ satisfying the equation $\mathrm{tg}\,(1/2\gamma\ln 2) = \gamma$, i.e., for $\mathrm{Re} \approx 7.67$, the denominator of the right-hand side of (2.6) vanishes at $x = 1$. When that happens the root and pole coincide at the point $x = 1$, the number $y(1)$ becomes finite and by (2.6) it is equal to 4. The limiting external value of $y_*(x)$ is analytic, but its physical meaning is a sink of strength $8\pi\nu$ uniformly distributed along the axis.

In the near-critical situation a narrow internal region of high velocities is formed around the axis - a strong jet zone. In this zone y falls from nearly 4 at the boundary to zero on the axis, while y' takes large negative values.

We introduce a small parameter ε and a new independent variable η:

$$\varepsilon = -1/y'(1), \quad \eta = (1 - x)/\varepsilon \tag{2.7}$$

Using the relationship $y_x' = -y_\eta'/\varepsilon$, we obtain from Eq.(2.5)

$$-\eta\,(2 - \varepsilon\eta)\,y_\eta' + 2\,(1 - \varepsilon\eta)\,y = {}^1/_2 y^2 - \varepsilon^2\eta^2 C_1$$

Letting ε go to zero, we obtain an equation for the principal term of the expansion of the axial boundary layer:

$$\eta y_\eta^{*\prime} = y^*\,(1 - y^*/4), \quad y^*(0) = 0 \tag{2.8}$$

which yields Schlichting's solution $y^* = 4\eta/(4 + \eta)$. /9/.

It should be noted that this boundary-layer solution corresponds to an internal expansion and does not depend on C_1. Since the only essential conditions for Schlichting's solution are regularity on the axis and the existence of a boundary layer, it approximates a broad range of jet-type flows near the axis. Returning to the original variables, we can write it as

$$y^* (x) = -4y' (1) (1 - x)/[4 - y' (1) (1 - x)] \tag{2.9}$$

As an external expansion $y_* (x)$ in this problem one obtains the solution (2.6) itself in the critical situation.

Thus, this example, considered together with Squire's solution, clearly demonstrates the formation of an extremely strong jet and a crisis of flow at finite Reynolds numbers. Moreover, this crisis is unremovable. In physical terms, it means that the nucleus of the jet will always become turbulent.

If one assumes adherence conditions $y (0) = y' (0) = 0$, with $y' (1)$ bounded, the external solution $y_* (x)$ in the critical case satisfies the condition $y_* (1) = 4$ and the equation /10/

$$(1 - x^2) y_*' + 2xy_* = {}^1/_2 y_*^2 + C_1 x (1 - x) \tag{2.10}$$

The value $C_1 \approx 15.29$ is found by integration from the axis using the condition $y_* (0) = 0$. Graphically, y_* can be approximated by a cubic polynomial whose coefficients are determined by the boundary conditions:

$$y_* \approx x^2 [4 + (6 - C_1/4) (1 - x)] \tag{2.11}$$

3. *Crisis in the "waterspout" problem.* The interaction of a vortex filament with a plane is described in the context of the selfsimilar class (1.1) and reduces /1, 2/ to the solution of a boundary value problem for Eqs.(1.3), (2.1), (2.2), assuming adherence conditions on the plane and regularity of the meridian velocity field at the axis for a given circulation. It has been proved /1, 2/ that if the Reynolds number $\mathrm{Re} = \Gamma_1/\nu$ exceeds the critical value $\mathrm{Re}_* \approx 5.53$, there is no longer a selfsimilar solution satisfying these conditions. Below, we shall investigate the structure of the solutions in a neighbourhood of $\mathrm{Re} = \mathrm{Re}_* - \varepsilon_1$, $0 < \varepsilon_1 \ll 1$, subsequently considering the problem when the circulation is given on a finite-angled cone. If the angle is zero the problem reduces /2/ to Eq.(1.3) with boundary conditions $\Gamma (0) = 0$, $\Gamma (1) = \mathrm{Re}$ and the equations

$$(1 - x^2) y' + 2xy - {}^1/_2 y^2 = {}^1/_2 \mathrm{Re}^2 x (1 - x) - G (x) \tag{3.1}$$

$$G (x) = (1 - x)^2 \int_0^x \frac{t\Gamma^2}{(1 - t^2)^2} dt + 2x \int_x^1 \frac{\Gamma^2}{(1 + t)^2} dt \tag{3.2}$$

$$y (0) = y' (0) = y (1) = 0$$

As $\mathrm{Re} \to \mathrm{Re}_*$ the circulation tends to zero everywhere except on the axis itself /2/ - this is also illustrated by numerical calculations /11/. It then follows directly from (3.2) that $G (x) \to 0$ as $\mathrm{Re} \to \mathrm{Re}_*$. Hence, as far as the principal term of the external expansion with respect to ε_1 is concerned, the function $G (x)$ on the right of Eq.(3.1) may be ignored, and Eq.(3.1) reduces to (2.10) with the renamed constant $C_1 = \mathrm{Re}_*^2/2$. This implies a relationship between the critical Reynolds number and the value of the constant C_1 in (2.10): $\mathrm{Re}_* = \sqrt{2C_1} \approx 5.53$. Thus, the external expansion reduces to an expression describing the meridian flow generated by the sink along the axis, which is precisely the solution $y_* (x)$.

As for the internal expansion, it can be shown (see /2/) that the right-hand side of (3.1) has a zero of more than the first order at $x = 1$, so that here too the equation for the meridian motion splits when one is determining the principal terms. The principal term turns out to be the solution of (2.9) (the Schlichting jet). To obtain the boundary-layer solution for the circulation it is again convenient to use the small parameter $\varepsilon = -1/y' (1)$ and the variable $\eta = (1 - x)/\varepsilon$. Then, from (2.9) and (1.3), using the boundary conditions $\Gamma (\infty) = 0$, $\Gamma (0) = \mathrm{Re}$, one obtains

$$\Gamma^* = 4\mathrm{Re}/(4 + \eta) = 4\mathrm{Re}/[4 - y' (1) (1 - x)] \tag{3.3}$$
$$\Gamma_x^{*\prime} = -4\mathrm{Re}\, y' (1)/[4 - y' (1) (1 - x)]^2$$

Since the external expansion for the circulation is trivial: $\Gamma_* (x) \equiv 0$, the internal expansion can be continued up to the wall $x = 0$, at the same time satisfying the corrected boundary condition $\Gamma = 0$ at $x = 0$. Then $\Gamma^* = 4\mathrm{Re}\, x/[4 - y' (1) (1 - x)]$.

The small parameter ε vanishes together with $\varepsilon_1 = \mathrm{Re}_* - \mathrm{Re}$. Indeed, at $\mathrm{Re} = \mathrm{Re}_*$ the pole of $y (x)$ passes through the point $x = 1$, and therefore for small ε_1 the position of the pole x_p is approximately determined by the condition $x_p = 1 + A\varepsilon_1$, $A > 0$. On the other hand, the solution y^* defined by (2.9) may be rewritten as $y^* = 4 (1 - x)/[1 - x - 4/y' (1)]$, the position of the pole $x_p^* = 1 - 4/y' (1)$ for small ε_1 must coincide with x_p, whence it follows that $y' (1) = 4/[A (\mathrm{Re} - \mathrm{Re}_*)] + O (1)$.

Fig.1 illustrates the results obained by integrating equations (1.3), (3.1) and (3.2) at $\mathrm{Re} = 5.47$, which is near the critical value; in this case $y'(1) = -460.5$. (According to the data of /9/, in the case of a developed axisymmetric turbulent jet the turbulent viscosity ν_T develops in such a way that $rv_r/\nu_T = -y'(1) = 460.5$.) The figure compares the numerical results (the solid lines) with the analytic solutions (the dashed lines). Curves *1* and *2* represent the internal solution y^*, Γ^* and the external solution y_* respectively (see (2.9), (3.3), (2.11), and curve *3* the uniform asymptotic approximation $y_a = y^* + y_* - 4$. With increased distance from the axis the circulation decreases more slowly than in the boundary-layer approximation (3.3). This is due to the discrepancy between y^* and y far from the axis.

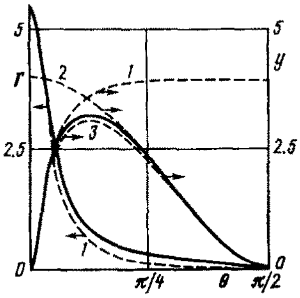

Fig.1

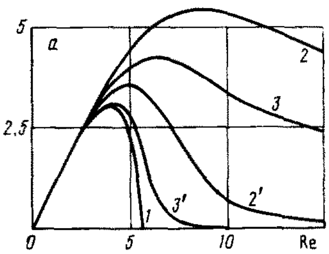

Fig.2

A good measure of the intensity of rotational motion near the plane is the quantity $a \equiv \Gamma'(0)$. The function $a(\mathrm{Re})$ (Curve *1* in Fig.2) is not monotone. As the circulation of the vortex filament increases, the rotation near the plane first becomes stronger and subsequently decays, finally vanishing at $\mathrm{Re} = \mathrm{Re}_*$. Computations indicate that the derivative $da/d\,\mathrm{Re}$ is finite at $\mathrm{Re} = \mathrm{Re}_*$ - its approximate value is -6.2.

This non-monotonic behaviour is due to the competition between two mechanisms of vorticity transfer - viscous diffusion and convection. For small Re values diffusion predominates, and therefore an increase in circulation (Re) on the axis causes an increase in circulation throughout the flow region. As Re increases, the drift of liquid towards the axis and the resulting inverse transfer of vorticity by convection are intensified. At Re values near Re_* the convective mechanism prevails, the circulation concentrates near the axis but diminishes far from the axis. Finally, at $\mathrm{Re} = \mathrm{Re}_*$ one has a catastrophe: irrespective of the presence of circulation on the axis, the liquid outside the axis does not rotate: a "collapse" of circulation occurs. But this is not the case for the pressure field generated by the vortex filament. The rarefaction on the axis creates a sink whose interaction with the plane generates an extremely intense jet.

4. A "waterspout" with a conical nucleus. The paradoxical situation in which the solution of the Navier-Stokes equations ceases to exist at a finite Reynolds number occurs when there is a singularity along the axis in the form of a vortex filament. The question arises, how does the situation change if the axis is excluded by imposing suitable boundary conditions on a finite-angled cone? The problem reduces to solving the system of Eqs.(1.3), (2.1) and (2.2) with adherence conditions on the plane, $y(0) = y'(0) = \Gamma(0) = 0$. On the cone $x = x_1$ one imposes the impermeability condition $y(x_1) = 0$ and the circulation is given: $\Gamma(x_1) = \mathrm{Re}$. A sixth boundary condition was imposed in two versions: adherence $(y'(x_1) = 0)$ and slip $(y''(x_1) = 0)$.

In numerical computations the integration was carried out from the wall to the axis, with the function F satisfying the conditions $F(0) = F'(0) = F''(0) = 0$. Then, by (2.2) and the adherence conditions at the wall, $C_3 = 0$. For the integration one also needs the values of C_1, C_2 and $a \equiv \Gamma'(0)$. Two of these three parameters must be chosen so as to satisfy two conditions on the cone. One of these conditions is $y(x_1) = 0$, the other, in the case of adherence, is $F(x_1) + C_2 x_1 - C_1 x_1^2 = 0$ by (2.2), and in the case of slip it is $F'(x_1) + C_2 - 2C_1 x_1 = 0$. The remaining parameter is a measure of the intensity of the flow, playing the role generally given to the Reynolds number. The Reynolds number itself is found after the problem has been solved: $\mathrm{Re} = \Gamma(x_1)$. It is convenient to leave free some parameter that depends monotonically on Re. As is clear from Fig.2, the parameter a is not appropriate in this sense; either of the parameters C_1, C_2 will do.

Figs.2-4 present the results of the computations. (The curves correspond to the following values: Curve *1* - $x_1 = 1$; Curve *2* - $x_1 = 0.99$, curve *3* - $x_1 = 0.999$, all with adherence conditions; Curves *2', 3'* - the same figures but with slip conditions). First, the results indicate that if the angle of the cone is finite, a solution exists at all Reynolds numbers, regardless of whether the conditions at the cone are adherence or slip. Second, here too, owing to the competition between diffusive and convective transfer of vorticity, the intensity of rotational motion near the plane is a non-monotone function of the circulation at the cone. As the Reynolds number increases, the rotation is first intensified and then weakens. However, unlike the case of a vortex filament, the rotation does not disappear at finite Reynolds number, tending only asymptotically to zero as the Reynolds number goes to infinity.

If the Reynolds number is fixed and the angle of the cone is decreased, the limiting solution at $\mathrm{Re} < \mathrm{Re}_*$ is that corresponding to a vortex filament. By (3.1), in this case $C_1 = C_2 = \mathrm{Re}^2/2$ (Curve *1* in Fig.3). When $\mathrm{Re} \geqslant \mathrm{Re}_*$, and $x_1 \to 1$, one has $a \to 0$; $C_1, C_2 \to \mathrm{Re}_*^2/2$. Outside the boundary layer the rotational motion disappears and the meridian flow becomes the same as at $\mathrm{Re} = \mathrm{Re}_*$.

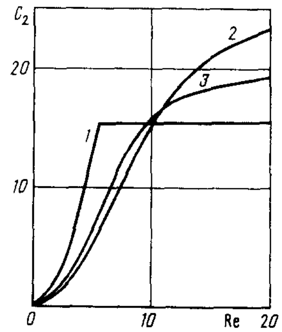

Fig.3

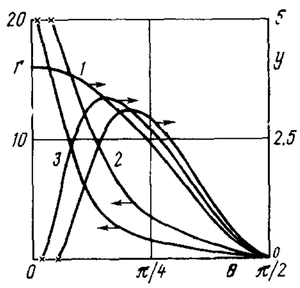

Fig.4

As is evident from Fig.4, which shows the results of computations carried out for $\mathrm{Re} = 20$, $y(x)$ at supercritical Reynolds numbers tends to $y_*(x)$ (Curve *1*) non-monotonically. The limiting solution $x_1 \to 1$ stabilizes very slowly, though this occurs more rapidly in the case of slip conditions (Fig.2).

In view of the results for these limiting cases, it may be assumed that the solution of the vortex filament problem can be continued into the region $\mathrm{Re} > \mathrm{Re}_*$, to which end one must put $a = 0$, $C_1 = C_2 = \mathrm{Re}_*^2/2$. However, the boundary condition $y(1) = 0$ is "eroded" and must be replaced by the condition $y(1) = 4$. A sink thus forms along the vortex filament, of strength independent of the circulation - it is equal to Q_*.

5. *Discussion.* The paradox presented by the non-existence of a solution for a finite Reynolds number /1/ has been resolved here by regularization of the singularity along the axis of symmetry. If the vortex filament is replaced by a vortex cone a solution exists for arbitrary Reynolds numbers. Since a vortex filament is essentially an idealization of real sources of flow, of finite dimensions, the approach proposed above is quite natural. The generalization proposed in /1/ for Serrin's formulation of the problem /3/ offers more possibilities of simulating atmospheric waterspout phenomena, though it is not essential for resolving the paradox.

Though the paradox has been eliminated there remains an interesting physical effect: focussing of rotational motion near the axis of symmetry and the formation of a strong rising jet. If a thin needle is rotated perpendicular to the plane, then at large distances compared with the radius of the needle the motion takes place in accordance with a selfsimilar solution. As the angular velocity of the needle is increased, the rotational motion of the liquid in the selfsimilar region will first intensify, but subsequently fall sharply to zero. At supercritical Reynolds numbers the rotational and jet motion is concentrated in the non-selfsimilar zone near the needle. In the selfsimilar region, however, the motion ceases to depend on the angular velocity of the needle and is the same as if it were generated by a sink of strength Q_* uniformly distributed along the axis.

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THE ASYMPTOTIC FORM OF THE UNEVENLY HEATED FREE BOUNDARY OF A CAPILLARY FLUID AT LARGE MARANGONI NUMBERS*

V.A. BATISHCHEV

Formal asymptotic expansions of the solution of the stationary problem of the thermocapillary flow of fluid in an unbounded region, with the free boundary unevenly heated, are constructed for large values of the Marangoni number. A non-linear boundary layer is formed near the free surface, and selfmodelling solutions are found for this layer near the critical point. A slow flow outside the boundary layer satisfies the equations of an ideal fluid. An equation describing the free boundary is obtained. When the temperature gradient vanishes, this equation becomes the well-known equation of the equilibrium of the free boundary of a capillary fluid. Numerical computations are carried out to determine the form of the meniscus at the vertical solid wall, the free boundary of the fluid poured onto a horizontal surface for the plane and axisymmetric case, and the surface of a gas bubble adjacent to the wall in a heated fluid.

The non-linear equations of the stationary boundary Marangoni layer near the free boundary of a fluid unevenly heated because of the thermocapillary effect were formulated in /1/ and studied earlier /2-6/. Asymptotic expansions of the solution of the stationary problem of a low-viscosity fluid flow under the action of tangential stresses were constructed in /7/.

1. Consider the stationary problem of the flow of an incompressible fluid in an unbounded region D under the action of thermocapillary forces caused by uneven heating of the free surface Γ, for the system of Navier-Stokes equations, with vanishing viscosity $\nu\to 0$

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g},\ \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

$$p = 2\nu\rho\mathbf{n}\cdot\Pi\cdot\mathbf{n} + \sigma(\varkappa_1+\varkappa_2) + p_*,\quad 2\nu\rho\Pi\cdot\mathbf{n} -$$
$$2\nu\rho\,(\mathbf{n}\cdot\Pi\cdot\mathbf{n})\,\mathbf{n} = \nabla_1\sigma,\ (x,y,z)\in\Gamma;\ \mathbf{v}\cdot\mathbf{n}\mid_\Gamma = 0,\ \mathbf{v}\mid_L = 0 \tag{1.2}$$

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